A topological aspect of the non-abelian anomaly for Weyl fermions on the lattice

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In the continuum a topological obstruction to the vanishing of the non-abelian anomaly in 2n dimensions is given by the index of a certain Dirac operator in 2n+2 dimensions. In this paper an analogous result is derived for Weyl fermions on the lattice in the Ginsparg–Wilson–Lüscher formulation.

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Recently there has been major progress on the long-standing problem of formulating chiral fermions on the lattice. It has been realised that the problems that arise for chirally symmetric lattice Dirac operators ($\gamma_5 D + D\gamma_5 = 0$), encapsulated in the Nielsen–Ninomiya no-go theorem [1], can be avoided by instead considering operators which satisfy the Ginsparg–Wilson relation [2]:

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D \tag{1}$$

where a=lattice spacing. For such operators, instead of $-\gamma_5 D = D\gamma_5$ one has $-\gamma_5 D = D\hat{\gamma_5}$ where $\hat{\gamma_5} = \gamma_5(1-aD)$. It follows from (1) that $\hat{\gamma}_5^2 = 1$, so $\hat{P}_\pm = \frac{1}{2}(1 \pm \hat{\gamma_5})$ are projection operators and we have $P_\mp D = D\hat{P}_\pm$ (where $P_\pm = \frac{1}{2}(1 \pm \gamma_5)$ are the usual chiral projections). The fermion action then decomposes into left- and right-handed parts [3,4]:

$$\bar{\psi} D\psi = \bar{\psi}_+ D\psi_- + \bar{\psi}_- D\psi_+ \tag{2}$$

where $\bar{\psi}_\pm = \psi P_\pm$, $\psi_\pm = \hat{P}_\pm \psi$. What makes this of interest is that acceptable lattice operators satisfying (1), which are not afflicted with the usual species doubling problem [1], have recently been found: Neuberger's overlap-Dirac operator [5] and generalisations thereof [6], and the operator in the so-called perfect actions [7]. Although solutions to (1) are never ultra-local [8], at least one solution − the overlap-Dirac operator − has been shown to be local [9]. Acceptable solutions must be gauge-covariant, i.e.

$$g \cdot (D^U \psi) = D^{g \cdot U}(g \cdot \psi) \tag{3}$$

where g is a gauge transformation acting on the lattice gauge field U and spinor field ψ , then the action (2) is invariant under chiral gauge transformations. In addition, a hermiticity condition is imposed:

$$(D^U)^* = \gamma_5 D^U \gamma_5 \tag{4}$$

(see, e.g., [3] for more background).

The construction of the quantum theory for this formulation of chiral fermions on the lattice has recently been discussed by M. Lüscher [10,11]. Taking the lattice to be finite so that the space $\mathcal{C}$ of lattice spinor fields is finite-dimensional, we set

$$\mathcal{C}_\pm = P_\pm(\mathcal{C}) \qquad , \quad \hat{\mathcal{C}}_\pm = \hat{P}_\pm(\mathcal{C}) \tag{5}$$

In the case of right-handed Weyl fermions the non-perturbative construction of the quantum theory in the path integral framework requires integration measures on $\mathcal{C}_-$ and $\hat{\mathcal{C}}_+$. These can be specified via unit volume elements v on $\mathcal{C}_-$ and $\hat{w}$ on $\hat{\mathcal{C}}_+$. Note that $\hat{\mathcal{C}}_+ = \hat{\mathcal{C}}_+(U)$ depends on the gauge field U since $D = D^U$ appears in the definition of $\hat{\gamma_5}$, $\hat{P}_\pm$. Therefore the unit volume element $\hat{w} = \hat{w}(U)$ depends on U. The key issue is whether there exists a smooth family $\{\hat{w}(U)\}$ such that the resulting quantum theory is free of gauge anomalies. In particular, the resulting effective action should be gauge-invariant. The effective action $\Gamma(U)$ is given by

$$e^{-\Gamma(U)} = \det D^U_{v,\hat{w}(U)} \tag{6}$$

where $\det D_{v,\hat{w}}$ is the proportionality constant between $D^U \hat{w}(U)$ and v :

$$D\hat{w} = (\det D_{v,\hat{w}})\, v \qquad , \qquad \det D_{v,\hat{w}} = \langle v \,, D\hat{w} \rangle \tag{7}$$

(recall that $D\hat{w} = D\hat{w}_1 \wedge \ldots \wedge D\hat{w}_d$ for any basis $\hat{w}_1, \ldots, \hat{w}_d$ for $\hat{\mathcal{C}}_+$ with $\hat{w}_1 \wedge \ldots \wedge \hat{w}_d = \hat{w}$). In the abelian case [10] Lüscher showed that a smooth family $\{\hat{w}(U)\}$ for which the effective action (6) is gauge-invariant does indeed exist [10,12] (see also [13]). However, the existence of such a family in the non-abelian case [11] is still an open question. In this paper we show that a necessary condition for its existence in 2n dimensions is the vanishing of the index of a certain "Dirac" operator in 2n+2 dimensions. This result is a lattice analogue of the main result in the classic paper [14] (also obtained in [15] in a more general mathematical setting). The relationship between the Dirac operator index and its density (the abelian anomaly [16]) in 2n+2 dimensions and the non-abelian anomaly in 2n dimensions is a central feature of the anomaly structure of continuum quantum gauge theories (see, e.g., [17,14]). This paper is a first step towards establishing analogues of these structures in the present lattice setting. (The non-abelian anomaly in the overlap formalism was recently discussed in [18], although no connection with a Dirac operator in 2n+2 dimensions was made there.)

Let D^U be a lattice operator on $\mathcal{C}$ satisfying the GW relation (1) and (3)–(4). Although D is not chirally symmetric, $\ker D$ is invariant under γ_5 so index$D$ is well-defined [19]. Let $\mathcal{U}$ denote the space of lattice gauge fields for which (i) D^U is defined (e.g. the overlap-Dirac operator is defined for all U away from a certain lower-dimensional manifold) and (ii) index$D^U = 0$ (since it is only in this case that the chiral determinant (6) is non-trivial, c.f. [11]). If $\hat{w}(U)$ is a unit volume element on $\hat{\mathcal{C}}_+(U)$ then any other unit volume element $\hat{w}_1(U)$ differs from it by a phase factor $e^{i\theta(U)}$, so the same is true for the determinants:

$$\det D^U_{v,\hat{w}_1(U)} = e^{i\theta(U)} \det D^U_{v,\hat{w}(U)} \tag{8}$$

The modulus of $\det D^U_{v,\hat{w}(U)}$ is independent of $\hat{w}(U)$ and gauge-invariant due to (3), so the potential gauge non-invariance arises only in the phase. Consider a family of lattice gauge transformations $g = \{g_\theta\}_{\theta\in[0,2\pi]}$ with $g_{2\pi} = g_0$. For each $U \in \mathcal{U}$ it determines a map

$$S^1 \to \mathcal{U} \qquad\qquad \theta \mapsto g_\theta \cdot U \tag{9}$$

For $U \in \mathcal{U}$ with $\det D^U_{v,\hat{w}(U)} \ne 0$ consider the map $S^1 \to \mathbf{C} - \{0\}$ given by

$$\theta \mapsto \det D_\theta := \det D^{g_\theta\cdot U}_{v,\hat{w}(g_\theta\cdot U)} \tag{10}$$

The winding number (degree) $d(g,U)$ of this map is independent of the choice of v and $\hat{w}(U)$ –this is clear from (8)– and is thus a topological obstruction to constructing a gauge-invariant chiral determinant. We are going to show that $d(g,U)$ is the index of a certain "Dirac" operator in 2n+2 dimensions. To this end we choose an extension of (9) to a map

$$B^2 \to \mathcal{U} \qquad\qquad (\theta,t) \mapsto U(\theta,t) \tag{11}$$

where B^2 is the disc with polar coordinates (θ,t) and $U(\theta,1) = g_\theta \cdot U$. (We assume that such an extension exists for the g and U under consideration. Whether it exists in general depends on the topological structure of $\mathcal{U}$. But we can at least say that one exists when considering the classical continuum limit: an extension is easily constructed in the continuum [14] and we can then take its lattice transcript.) Then (10) extends to a map $B^2 \to \mathbf{C}$ given by

$$(\theta,t) \mapsto \det D_{(\theta,t)} := \det D^{U(\theta,t)}_{v,\hat{w}(U(\theta,t))} \tag{12}$$

Generically, the points in B^2 at which $\det D_{(\theta,t)}$ vanishes are isolated. $\det D_{(\theta,t)}$ has a winding number around each of these points, and their sum equals $d(g,U)$ (c.f. [14]).

$\det D_{(\theta,t)}$ can be considered as a tangent vector field on B^2 after identifying $\mathbf{C}$ with $\mathbf{R}^2$. The winding number around a vanishing point (θ_0,t_0) is then the index of the vector field at the point. Generically, the vanishing points of such fields are non-degenerate and the index of the vector field at such a point is ±1 with sign given by the sign of the determinant of the Jacobi matrix at the point (see, e.g., [20]). In the present case this determinant is

$$\det\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \mathrm{Im}(z_1^* z_2) \tag{13}$$

where $z_k = x_k + iy_k$ $(k=1,2)$ are given as follows. At the vanishing point (θ_0,t_0) $D^{U(\theta_0,t_0)}$ has a zero-mode w_0 in $\hat{\mathcal{C}}_+(U(\theta_0,t_0))$ (for simplicity we assume there is just one). Recalling index$D = 0$ and noting that $\hat{\gamma}_5 = \gamma_5$ on $\ker D$ it follows that $D^{U(\theta_0,t_0)}$ also has a (single) zero-mode v_0 in $\mathcal{C}_-$ and $\gamma_5 w_0 = w_0$, $\gamma_5 v_0 = -v_0$. Now let $q = (q_1,q_2)$ be

Euclidean coordinates on the disc B^2. In the following q will be considered as a function of θ, t and vice versa, and we will write $U(q)$ in place of $U(\theta,t)$. The z_k's in (13) are given by

$$z_k \;\equiv\; \langle v_0\,,(\tfrac{\partial}{\partial q_k}D^{U(q)})_{q=q^{(0)}}w_0\rangle \qquad k=1,2 \tag{14}$$

where $q^{(0)}=q(\theta_0,t_0)$ and the winding number/index at $q^{(0)}$ is ± 1 with sign given by that of (13). (A derivation of this is given in the Appendix. Practically the same result was found in the continuum setting in [14].) Henceforth all vanishing points $q^{(0)}$ are assumed to be non-degenerate. Then it is easy to see that near each $q^{(0)}$ the coordinates (q_1,q_2) can be chosen such that

$$\mathrm{Re}(z_1^*z_2)=0 \tag{15}$$

We fix a choice of coordinate system satisfying this, to be used in the following.

We now construct the "Dirac" operator in 2n+2 dimensions. In 2n+2 dimensions the gamma matrices for the Clifford algebra can be chosen as

$$\Gamma^\mu=\sigma_1\otimes\gamma^\mu \;\;(\mu=1,\dots,2n)\;,\;\; \Gamma^{2n+1}=\sigma_2\otimes 1\;,\;\; \Gamma^{2n+2}=\sigma_1\otimes\gamma_5 \tag{16}$$

acting on $\mathbf{C}^2\otimes\mathbf{C}^{2^n}$, where $\{\sigma_j\}_{j=1,2,3}$ are the Pauli matrices and $\{\gamma^\mu\}$ are the gamma matrices in 2n dimensions. The corresponding chirality matrix Γ_5 satisfying $\Gamma_5\Gamma^\alpha=-\Gamma^\alpha\Gamma_5$ and $\Gamma_5^2=1$ is

$$\Gamma_5 \;\equiv\; i^{n+1}\prod_{\alpha=1}^{2n+2}\Gamma^\alpha=\sigma_3\otimes 1 \tag{17}$$

Let $\mathcal{S}$ denote the vector space of pairs $(\Psi,\widetilde{\Psi})$ where $\Psi(x,\theta,t)$ is a linear combination of elements of the form $f(\theta,t)\otimes\psi(x)$ for smooth functions $f:B^2\to\mathbf{C}^2$ and elements $\psi\in\mathcal{C}$, $\widetilde{\Psi}(x,\theta,s)$ is similarly a linear combination of $\widetilde{f}(\theta,s)\otimes\psi(x)$ where $\widetilde{f}:\widetilde{B}^2\to\mathbf{C}^2$ is now defined on another copy of the disc, and moreover Ψ , $\widetilde{\Psi}$ satisfy

$$\widetilde{\Psi}(x,\theta,1)=(1\otimes g_\theta(x))\Psi(x,\theta,1) \tag{18}$$

$\mathcal{S}$ can be considered as the space of sections in a vector bundle over $S^2=B^2\cup_{S^1}\widetilde{B}^2$ with fibre $\mathbf{C}^2\otimes\mathcal{C}$ and structure group $1\otimes\mathcal{G}$, where $\mathcal{G}$ is the group of gauge transformations on $\mathcal{C}$. For $(\Psi,\widetilde{\Psi})\in\mathcal{S}$ define $(\mathcal{D}\Psi,\widetilde{\mathcal{D}}\widetilde{\Psi})\in\mathcal{S}$ by

$$(\mathcal{D}\Psi)(x,\theta,t)=\Big(\sigma_1\otimes iD^{U(\theta,t)}+\sum_{\alpha=1,2}\Gamma^{2n+\alpha}i\partial_\alpha\Big)\Psi(x,\theta,t) \tag{19}$$

$$(\widetilde{\mathcal{D}}\widetilde{\Psi})(x,\theta,s)=\Big(\sigma_1\otimes iD^{U}+\sum_{\alpha=1,2}\Gamma^{2n+\alpha}i(\partial_\alpha+sg_\theta(x)^{-1}\partial_\alpha g_\theta(x))\Big)\widetilde{\Psi}(x,\theta,s) \tag{20}$$

where $\partial_1\equiv\frac{\partial}{\partial q_1}$, $\partial_2\equiv\frac{\partial}{\partial q_2}$. It is easy to check that $(\mathcal{D}\Psi,\widetilde{\mathcal{D}}\widetilde{\Psi})$ satisfies (18) and we have hereby defined a linear operator $\underline{\mathcal{D}}=(\mathcal{D},\widetilde{\mathcal{D}})$ on $\mathcal{S}$. Note that $\underline{\mathcal{D}}\ne\underline{\mathcal{D}}^*$, but that $\underline{\mathcal{D}}$ and $\underline{\mathcal{D}}^*$ both anticommute with Γ_5. The self-adjoint, chirally symmetric operator $\frac{1}{2}(\underline{\mathcal{D}}+\underline{\mathcal{D}}^*)$ is an analogue of the Dirac operator in 2n+2 dimensions introduced in [14]. (The continuum operator can be obtained by replacing $D^{U(\theta,t)}$ by $\not{\partial}_{A_{(t,\theta)}}$ in (19) and D^U by $\not{\partial}_A$ in (20) and using (16).) It has a well-defined index $\equiv$ $\mathrm{Tr}(\Gamma_5|_{\ker(\underline{\mathcal{D}}+\underline{\mathcal{D}}^*)})$ and we have the following analogue of the main result in [14]:
Theorem

$$d(g,U)=\mathrm{index}(\underline{\mathcal{D}}+\underline{\mathcal{D}}^*) \tag{21}$$

The remainder of this paper is concerned with the derivation of (21). Our derivation is modelled on the continuum one in [14], although additional complications arise here due to the fact that D is not chirally symmetric and $\underline{\mathcal{D}}\ne\underline{\mathcal{D}}^*$. A deformation $\underline{\mathcal{D}}_\epsilon=(\mathcal{D}_\epsilon,\widetilde{\mathcal{D}}_\epsilon)$ of $\underline{\mathcal{D}}$ is defined for $\epsilon>0$ by replacing $\sigma_1\otimes iD^{U(\theta,t)}$ by $\frac{1}{\epsilon}\sigma_1\otimes iD^{U(\theta,t)}$ in (19) and $\sigma_1\otimes iD^U$ by $\frac{1}{\epsilon}\sigma_1\otimes iD^U$ in (20). Since the index is invariant under continuous deformations we have

$$\mathrm{index}(\underline{\mathcal{D}}+\underline{\mathcal{D}}^*)=\mathrm{index}(\underline{\mathcal{D}}_\epsilon+\underline{\mathcal{D}}_\epsilon^*)\qquad\forall\epsilon>0$$

We will derive a one-to-one correspondence between the zero-modes of

$$\underline{\mathcal{H}}_\epsilon=\frac{1}{2}(\underline{\mathcal{D}}_\epsilon^*\,\underline{\mathcal{D}}_\epsilon+\underline{\mathcal{D}}_\epsilon\,\underline{\mathcal{D}}_\epsilon^*) \tag{22}$$

in the $\epsilon \to 0$ limit and the zero-modes of $\{D^{U(q)}\}_{q\in B^2}$, with the Γ_5-chiralities of the former coinciding with the winding numbers associated with the latter. Noting that $\ker \underline{\mathcal{H}}_\epsilon = \ker(\underline{\mathcal{D}}_\epsilon) \cap \ker(\underline{\mathcal{D}}_\epsilon^*)$ the theorem then follows from the following

Lemma.

$$\mathrm{index}(\underline{\mathcal{D}}_\epsilon + \underline{\mathcal{D}}_\epsilon^*) = \mathrm{Tr}(\Gamma_5|_{\ker(\underline{\mathcal{D}}_\epsilon)\cap\ker(\underline{\mathcal{D}}_\epsilon^*)})\,. \tag{23}$$

Proof of the lemma. Set $\mathcal{V}_\epsilon = \ker(\underline{\mathcal{D}}_\epsilon + \underline{\mathcal{D}}_\epsilon^*)$. We show below that $\underline{\mathcal{D}}_\epsilon(\mathcal{V}_\epsilon) \subseteq \mathcal{V}_\epsilon$. Then $i\underline{\mathcal{D}}_\epsilon$ is a self-adjoint operator on $\mathcal{V}_\epsilon$ and since it anticommutes with Γ_5 its eigenvectors with non-zero eigenvalues come in $\pm$ Γ_5-chirality pairs, so

$$\mathrm{index}(\underline{\mathcal{D}}_\epsilon + \underline{\mathcal{D}}_\epsilon^*) \;\equiv\; \mathrm{Tr}(\Gamma_5|_{\mathcal{V}_\epsilon}) = \mathrm{Tr}(\Gamma_5|_{\ker\{i\underline{\mathcal{D}}_\epsilon:\mathcal{V}_\epsilon\to\mathcal{V}_\epsilon\}})$$

The lemma then follows from the fact that $\ker\{i\underline{\mathcal{D}}_\epsilon : \mathcal{V}_\epsilon \to \mathcal{V}_\epsilon\} = \ker(\underline{\mathcal{D}}_\epsilon) \cap \ker(\underline{\mathcal{D}}_\epsilon^*)$. It remains to show $\underline{\mathcal{D}}_\epsilon(\mathcal{V}_\epsilon) \subseteq \mathcal{V}_\epsilon$. For this we note the following characterisation of $\mathcal{V}_\epsilon$: It follows from (1) and (4) that $DD^* = D^*D$, i.e. D is normal and therefore has a basis of eigenvectors:

$$D\psi_m = \lambda_m\psi_m \qquad , \qquad D^*\psi_m = \lambda_m^*\psi_m \tag{24}$$

Now if $\underline{\Psi} = (\Psi, \widetilde{\Psi}) \in \mathcal{S}$ then $\Psi\,, \widetilde{\Psi}$ can be written as

$$\Psi(x,\theta,t) = \sum_m f_m(\theta,t)\otimes\psi_m(x) \qquad\qquad \widetilde{\Psi}(x,\theta,s) = \sum_l \widetilde{f}_l(\theta,s)\otimes\psi_l(x) \tag{25}$$

It is easy to check that $\underline{\Psi} \in \mathcal{V}_\epsilon$, i.e. $\underline{\mathcal{D}}_\epsilon(\underline{\Psi}) = -\underline{\mathcal{D}}_\epsilon^*(\underline{\Psi})$, precisely when the eigenvalues λ_m, λ_l of the ψ_m's and ψ_l's in (25) are all real. Clearly if $\underline{\Psi}$ satisfies this criterion then so does $\underline{\mathcal{D}}_\epsilon(\underline{\Psi})$. This completes the proof of the lemma.

We now turn to the operator $\underline{\mathcal{H}}_\epsilon = (\mathcal{H}_\epsilon, \widetilde{\mathcal{H}}_\epsilon)$ in (22) and relate its zero-modes in the $\epsilon \to 0$ limit to those of $\{D^{U(q)}\}$ as promised. Our argument is somewhat heuristic, at the same level of rigour as the one in [14]. A calculation gives

$$\mathcal{H}_\epsilon = \frac{1}{2}(\mathcal{D}_\epsilon^*\mathcal{D}_\epsilon + \mathcal{D}_\epsilon\mathcal{D}_\epsilon^*) = \frac{1}{\epsilon^2}1\otimes D^*D - 1\otimes\sum_{\alpha=1,2}\partial_\alpha^2 + \frac{1}{\epsilon}\Big(\sigma_3\otimes\tfrac{1}{2}i(\partial_1 D - \partial_2 D^*) + 1\otimes i\gamma_5\tfrac{1}{2}i(\partial_2 D - \partial_2 D^*)\Big) \tag{26}$$

where $D = D^{U(\theta,t)} = D^{U(q)}$ and we have used $D^*D = DD^*$. Because the leading order term in the $\epsilon \to 0$ limit is $\frac{1}{\epsilon^2}1\otimes(D^{U(q)})^*D^{U(q)}$ the zero-modes of $\mathcal{H}_\epsilon$ are localised around zero-modes of $\{D^{U(q)}\}_{q\in B^2}$. On the other hand, the leading order term in $\widetilde{\mathcal{H}}_\epsilon = \frac{1}{2}(\widetilde{\mathcal{D}}_\epsilon^*\widetilde{\mathcal{D}}_\epsilon + \widetilde{\mathcal{D}}_\epsilon\widetilde{\mathcal{D}}_\epsilon^*)$ in the $\epsilon \to 0$ limit is $\frac{1}{\epsilon^2}1\otimes(D^U)^*D^U$. Since by assumption D^U has no zero-modes we see that the zero-modes of $\underline{\mathcal{H}}_\epsilon$ in the $\epsilon \to 0$ limit are of the form $(\Psi, 0)$ where Ψ is a zero-mode of $\mathcal{H}_\epsilon$ localised around zero-modes of $D^{U(q)}$. Such Ψ will be a linear combination of of fields of the form

$$\Psi_{q^{(0)}}(x,q) = f_+(q)\otimes w_0 + f_-(q)\otimes v_0 \tag{27}$$

where $q^{(0)}$ is a point at which $D^{U(q)}$ has a zero-mode and $\{v_0, w_o\}$ is an orthonormal basis for $\ker D^{U(q^{(0)})}$ with $\gamma_5 v_0 = -v_0$ and $\gamma_5 w_0 = w_0$ as previously. If such a zero-mode for $\mathcal{H}_\epsilon$ exists the $\mathbf{C}^2$-valued functions $f_\pm(q)$ will be concentrated near $q^{(0)}$, so for the purpose of investigating the existence and properties of the zero-mode we can make the approximation

$$D^{U(q)} \;\simeq\; D^{U(q^{(0)})} + (\tfrac{\partial}{\partial q_1}D)\,\tilde{q}_1 + (\tfrac{\partial}{\partial q_2}D)\,\tilde{q}_2 \tag{28}$$

where $\tilde{q}_k = q_k - q_k^{(0)}$ and the derivatives are evaluated at $q^{(0)}$. Furthermore, we approximate the operators by their projections onto the subspace spanned by the fields of the form (27); they can then be written as 2x2 matrices acting on $(f_+(q), f_-(q))$. The diagonal elements for D vanish since

$$\langle w_0\,, \tfrac{\partial}{\partial q_k}Dw_0\rangle = \langle v_0\,, \tfrac{\partial}{\partial q_k}Dv_0\rangle = 0 \tag{29}$$

(this follows easily from applying $\frac{\partial}{\partial q_k}$ to the identities $\langle w_0\,, D^{U(q)}w_0(q)\rangle = 0$ and $\langle v_0\,, D^{U(q)}v_0\rangle = \langle v_0\,, D^{U(q)}\hat{P}_+v_0\rangle = \frac{-a}{2}\langle v_0\,, D^{U(q)}\gamma_5 D^{U(q)}v_0\rangle$; these and the following calculation are similar to the one leading to (A7) in the Appendix.) To find the off-diagonal elements we recall $\langle v_0\,, \frac{\partial}{\partial q_k}Dw_0\rangle \;\equiv\; z_k$ and calculate

$$\langle w_0\,, \tfrac{\partial}{\partial q_k}Dv_0\rangle = \tfrac{\partial}{\partial q_k}\,\langle w_0(q)\,, D^{U(q)}v_0\rangle = \tfrac{\partial}{\partial q_k}\,\langle\gamma_5 D^{U(q)}\gamma_5 w_0(q)\,, v_0\rangle = -\langle\tfrac{\partial}{\partial q_k}Dw_0\,, v_0\rangle = -z_k^* \tag{30}$$

Analogous relations for D^* can be obtained using $D^* = \gamma_5 D \gamma_5$, e.g.

$$\langle v_0 \, , \tfrac{\partial}{\partial q_k} D^* w_0 \rangle = -\langle v_0 \, , \tfrac{\partial}{\partial q_k} D w_0 \rangle = -z_k \, , \tag{31}$$

leading to

$$D^{U(q)} \;\to\; \begin{pmatrix} 0 & -z_k^* \tilde{q}_k \\ z_k \tilde{q}_k & 0 \end{pmatrix} \qquad , \qquad (D^{U(q)})^* \;\to\; \begin{pmatrix} 0 & z_k^* \tilde{q}_k \\ -z_k \tilde{q}_k & 0 \end{pmatrix} \tag{32}$$

where summation over $k\!=\!1,2$ is implied. Multiplying these and using (15) we get

$$(D^{U(q)})^* D^{U(q)} \;\to\; (|z_1|^2 \tilde{q}_1^2 + |z_2|^2 \tilde{q}_2^2)\, \mathbf{1} \tag{33}$$

The last term in the expression (26) for $\mathcal{H}_\epsilon$ is of lower order in $\frac{1}{\epsilon}$ than the first so we simply approximate it by its value at $q = q^{(0)}$ (i.e. higher order terms in $\tilde{q}_1, \tilde{q}_2$ are discarded). The corresponding 2x2 matrix is easily determined using (31). We can assume that the field in (27) has definite Γ_5-chirality $\chi\!=\!\pm 1$, then the approximation for (26) is

$$\mathcal{H}_\epsilon \to (-\frac{\partial^2}{\partial \tilde{q}_1^2} - \frac{\partial^2}{\partial \tilde{q}_2^2} + \frac{1}{\epsilon^2}(|z_1|^2 \tilde{q}_1^2 + |z_2|^2 \tilde{q}_2^2)) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{\epsilon} i \begin{pmatrix} 0 & \chi z_1^* + i z_2^* \\ -\chi z_1 + i z_2 & 0 \end{pmatrix} \tag{34}$$

This is precisely the operator found in [14] (after replacing $z_k \to i z_k$). As discussed there, its smallest eigenvalue (which has multiplicity=1) is

$$\frac{1}{\epsilon} \Big(|z_1| + |z_2| - \sqrt{|z_1|^2 + |z_2|^2 + 2\chi\, \mathrm{Im}(z_1^* z_2)} \Big) \tag{35}$$

so there is a zero-mode precisely when

$$|z_1||z_2| = \chi\, \mathrm{Im}(z_1^* z_2) \tag{36}$$

Note $|z_1||z_2| = |\mathrm{Im}(z_1^* z_2)|$ due to (15), so (36) reduces to

$$\mathrm{sign}(\chi) = \mathrm{sign}(\mathrm{Im}(z_1^* z_2)) \tag{37}$$

Recalling (13) we conclude that a zero-mode of $D^{U(q)}$ corresponds to a zero-mode of $\mathcal{H}_\epsilon$ in the $\epsilon \to 0$ limit, with the Γ_5-chirality of the latter equalling the winding number associated with the former. The converse can also be (heuristically) derived as in [14]. The theorem now follows from the previous lemma as discussed above.

Concluding remarks. To construct the effective action (6), $\hat{w}(U)$ does not actually need to be well-defined when D^U has a zero-mode. This opens the possibility that $\hat{w}(U)$ may have non-trivial topological structure. In particular, $\hat{w}(U(\theta,t))$ may have non-trivial winding number around a vanishing point (θ_0, t_0) for $\det D_{(\theta,t)}$. Thus one could attempt to construct $\hat{w}(U)$ such that these winding numbers always cancel those of $\det D_{(\theta,t)}$, thereby removing the topological obstruction to the gauge-invariance of the effective action.

Problems for future work: (1) The theorem should be rigorously proved. The corresponding continuum result in [14] can be rigorously obtained via the determinant line bundle results of Bismut and Freed [21] so one might attempt to carry these over to the present lattice setting. (2) The classical continuum limit should be calculated and shown to reproduce the continuum result. In the continuum the index of the Dirac operator in 2n+2 dimensions equals the degree of a certain map $\tilde{g} : S^{2n+1} \to G$ (where G is the gauge group) [14], leading to the condition $\pi_{2n+1}(G) = 1$ for anomaly-free theory. We should also see this condition emerging in the lattice setting. (3) Derive an analogue of the continuum relation [17,14] between the index *density* of the operator $\mathcal{D}$ in 2n+2 dimensions and the non-abelian anomaly [11] in 2n dimensions.

The non-abelian anomaly for Wilson and Kogut–Susskind fermions on the lattice was investigated in [22] and was seen to reproduce the continuum expression in the classical continuum limit. In these formulations the action is not invariant under chiral gauge transformations, while the path integral measure is invariant. In contrast, in the present Ginsparg–Wilson–Lüscher formulation it is the measure which is gauge non-invariant. It was shown in [11] that the (covariant) non-abelian anomaly in this setting reproduces the continuum expression in the classical continuum limit.

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APPENDIX

The Jacobi determinant (13), which determines the sign of the winding number ± 1 of $\det D_{(\theta,t)}$ around a vanishing point (θ_0, t_0), is obtained as follows. As in the text let $\{v_0, w_0\}$ be a basis for $\ker D^{U(\theta_0,t_0)}$ with $\gamma_5 v_0 = -v_0$, $\gamma_5 w_0 = w_0$. Choose a small disc $B^2_{(\theta_0,t_0)}$ centred at (θ_0, t_0) in which $\det D_{(\theta,t)}$ has no other vanishing points. Introduce polar coordinates (ϕ, r) on $B^2_{(\theta_0,t_0)}$ and set $D^{(\phi,r)} = D^{\tilde{U}(\phi,r)}$ with $\tilde{U}(\phi,r)$ specified by (ϕ,r) via the map $B^2 \to \mathcal{U}$ in (11). Our condition index$D = 0$ implies that $\hat{\mathcal{C}}_+$, $\hat{\mathcal{C}}_-$, $\mathcal{C}_+$, $\mathcal{C}_-$ all have the same dimension (c.f. §1 of [11]), so for $r > 0$ the restriction of $D^{(\phi,r)}$ to $D_+^{(\phi,r)} : \hat{\mathcal{C}}_+(\phi,r) \to \mathcal{C}_-$ is invertible. Define the unit vector $\widetilde{w}_0(\phi,r) \in \hat{\mathcal{C}}_+(\phi,r)$ by

$$\widetilde{w}_0(\phi,r) = (D_+^{(\phi,r)})^{-1}(v_0) \Big/ |(D_+^{(\phi,r)})^{-1}(v_0)| \qquad r > 0\,. \tag{A1}$$

Clearly in the limit $r \to 0$ $\widetilde{w}_0(\phi,r)$ coincides with w_0 up to a phase factor, i.e.

$$\lim_{r\to 0} \widetilde{w}_0(\phi,r) = e^{-i\beta(\phi)} w_0 \tag{A2}$$

Let $V_0 \subset \mathcal{C}_-$ and $W_0(\phi,r) \subset \hat{\mathcal{C}}_+(\phi,r)$ denote the one-dimensional subspaces spanned by v_0 and $\widetilde{w}_0(\phi,r)$ respectively. Then the induced map $(D_+^{(\phi,r)})^{-1} : \mathcal{C}_- \big/ V_0 \to \hat{\mathcal{C}}_+(\phi,r) \big/ W_0(\phi,r)$ defined for $r > 0$ extends to a well-defined map at $r = 0$ (with $W_0(\phi,r) \equiv \mathbf{C} \cdot w_0$ at $r = 0$). Now, choosing a volume element v_1 on $\mathcal{C}_- \big/ V_0$ we get a volume element $w_1(\phi,r) = (D_+^{(\phi,r)})^{-1} v_1$ on $\hat{\mathcal{C}}_+(\phi,r) \big/ W_0(\phi,r)$. It follows from (A2) that

$$w_0(\phi,r) = e^{i\beta(\phi)}\, \widetilde{w}_0(\phi,r) \tag{A3}$$

is well-defined and equal to w_0 at $r = 0$. Then

$$\hat{w}(\phi,r) = w_0(\phi,r) \wedge w_1(\phi,r) \tag{A4}$$

is a volume element on $\hat{\mathcal{C}}_+(\phi,r)$ well-defined at $r = 0$. Taking $v = v_0 \wedge v_1$ to be the volume element on $\mathcal{C}_-$ we find from (7) that

$$\det D^{(\phi,r)}_{v,\hat{w}(\phi,r)} = \langle v_0\,, D^{(\phi,r)} w_0(\phi,r) \rangle \tag{A5}$$

$$= |(D_+^{(\phi,r)})^{-1} v_0|^{-1}\, e^{i\beta(\phi)} \tag{A6}$$

($\hat{w}$ has not been normalised here, but this does not affect the winding number). It follows that the winding number of $\det D_{(\theta,t)}$ around the vanishing point (θ_0, t_0) is that of the phase factor $e^{i\beta(\phi)}$ determined in (A2). Moreover, going over to the Euclidean coordinates $q = (q_1, q_2)$, we see from (A5) that the winding number coincides with that of $\langle v_0\,, D^{U(q)} w_0(q) \rangle$ around $q^{(0)}$ (where we are writing $w_0(q)$ for $w_0(\phi,r)$). The Jacobi matrix in (13) now follows from

$$\tfrac{\partial}{\partial q_k}\big|_{q=q^{(0)}} \langle v_0\,, D^{U(q)} w_0(q) \rangle = \langle v_0\,, (\tfrac{\partial}{\partial q_k} D^{U(q)})_{q=q^{(0)}}\, w_0 \rangle = z_k \tag{A7}$$

which is derived using $D^* = \gamma_5 D \gamma_5$, $\gamma_5 v_0 = -v_0$, and $D^{U(q^{(0)})} v_0 = 0$.

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